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JHEP01(2009)035

(http://iopscience.iop.org/1126-6708/2009/01/035)

Published by IOP Publishing for SISSA
Received: *December 22, 2008*
Accepted: *December 29, 2008*
Published: *January 14, 2009*

Vector supersymmetry: Casimir operators and contraction from $\mathrm{OSp}(3, 2\,|2)$

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Abstract: We study some algebraic properties of the 'vector supersymmetry' (VSUSY) algebra, a graded extension of the four-dimensional Poincaré algebra with two odd generators, a vector and a scalar, and two central charges. The anticommutator between the two odd generators gives the four-momentum operator, from which the name vector supersymmetry. We construct the Casimir operators for this algebra and we show how both algebra and Casimirs can be derived by contraction from the simple orthosymplectic algebra $\mathrm{OSp}(3, 2\,|2)$. In particular, we construct the analogue of superspin for vector supersymmetry and we show that, due to the algebraic structure of the Casimirs, the multiplets are either doublets of spin $(s, s+1)$ or two spin $1/2$ states. Finally, we identify an odd operator, which is an invariant in a subclass of representations where a BPS-like algebraic relation between the mass and the values of the central charges is satisfied.

Keywords: Extended Supersymmetry, Space-Time Symmetries, Superspaces.

Contents

1. Introduction

The vector supersymmetry (VSUSY) algebra is a graded extension of the Poincaré algebra in four dimensions. Two fermionic operators are added, an odd Lorentz vector and an odd Lorentz scalar, and two central charges are allowed. The anticommutator between vector and scalar odd generators gives the four-momentum vector, from which the name vector supersymmetry.

This algebra was first introduced in [1] in 1976, with the purpose of obtaining a pseudo-classical description of the Dirac equation. However, to our knowledge its general algebraic properties have never been studied in detail. Due to the prominent role of supersymmetry algebras in field and string theories, VSUSY representations and possible realizations in physical models are worth exploring. In any case, it would be interesting to compare this alternative to ordinary supersymmetry to understand what the essential ingredients in supersymmetry are.

The main difference between vector and ordinary supersymmetry is that the odd generators of VSUSY do not satisfy the spin-statistics rule. This is not necessarily a problem for the construction of physical models with underlying VSUSY. In fact, a first example is the model for the spinning particle constructed in [2], where a quantization procedure preserving the underlying VSUSY has been applied, while the authors in [1] had to break the symmetry.[1]

There is also an interesting connection between VSUSY and topological field theories. In fact, an Euclidean version of VSUSY appears as a subalgebra of the symmetry algebra underlying topological $\mathcal{N} = 2$ Yang-Mills theories. Supersymmetry with odd vector generators was studied after Witten [4], who, in 1988, introduced topological $\mathcal{N} = 2$ Yang-Mills theories by performing a topological twist. After this twist, the fermionic generators become a vector, a scalar and an anti-selfdual tensor [5, 6]. After truncation of the anti-selfdual sector, the twisted algebra coincides with the Euclidean VSUSY algebra, in the special case when the two central charges of VSUSY are identified. Twisted topological algebras have proven to be useful in the study of renormalization properties of topological field theories [7, 8]. Moreover, a superspace formalism has been developed for these topological theories, see for example [5, 6, 9] and references therein.

An understanding of the physical content of theories with underlying VSUSY can be achieved by classifying VSUSY representations. A first step in this direction is to identify the Casimir operators of the algebra. We find that there are four Casimirs, P^2, Z, $\tilde{Z}$ and $\hat{W}^2$. Here P_μ is the four-momentum, Z and $\tilde{Z}$ are the central charges and $\hat{W}^2$ is the square of $\hat{W}_\mu$, the analogue of the superspin vector of ordinary supersymmetry [10], which is a generalization of the Pauli-Lubanski vector. The Pauli-Lubanski vector, which determines the spin of particles, can be written as a sum of this vector $\hat{W}_\mu$ and another one, $W_{C\,\mu}$. The latter is constructed in terms of the generators of the supertranslation subalgebra of VSUSY. It squares to P^2 and has spin $\frac{1}{2}$. As a result of this structure, a VSUSY irreducible representation contains two particles of Lorentz spin $s = |Y \pm \frac{1}{2}|$, where Y is the superspin having integer or half integer value. In particular, in the case $Y = 0$, one has two spin $1/2$ states. The described structure holds for the generic case with nonvanishing central charges, on which we focus in this paper. In the case of vanishing central charges the contribution of the Pauli-Lubanski vector vanishes in the expression of the superspin. We leave this special case for future work.

A comparison with the case of ordinary supersymmetry is in order. We observe that also in that case the Pauli-Lubanski vector can be written as the sum of the superspin vector and another spin vector (see for example [11]). The main difference between the two cases is that for ordinary supersymmetry the second spin vector does not square to a Casimir.

In case the mass and central charges satisfy a BPS-like relation, we identify an odd operator that behaves as a Casimir. In general, odd Casimirs can be present when the odd generators are scalars or vectors. In the case of VSUSY, there is no odd Casimir but we have constructed an odd nilpotent operator, invariant in the subclass of representations where the mentioned relation between mass and central charges is satisfied.

A good strategy to gain some understanding about a new algebra is to try to relate it to some other, better-known algebra. In this direction, we show that VSUSY arises as a contraction of the superalgebra OSp(3, 2 |2). OSp algebras are very simple generalizations

[1] See also [3].

of SO or Sp algebras. SO algebras have a symmetric metric, Sp algebras have an antisymmetric metric and OSp algebras have a 'graded symmetric' metric. OSp algebras are natural candidates for an embedding of VSUSY. The reason is that we need our VSUSY fermions to appear as vectors (or scalars) of the Lorentz group. In general, for OSp($M|N$) algebras the fermions are vectors of SO(N) and of Sp(M). We also need a bosonic factor to embed the two central charges. Therefore, we will use the embedding in OSp$(3,2\,|2)$, whose bosonic part is SO$(3,2) \times$ Sp(2). The latter factor can host the central charges we want to include.

We have also rederived the VSUSY Casimirs by contraction from OSp$(3,2\,|2)$. This procedure turns out to be rather nontrivial, mainly due to the fact that VSUSY has two central charges. In fact, OSp$(3,2\,|2)$ has three independent Casimirs, while, as mentioned above, VSUSY has four. Therefore, some nontrivial combination of the OSp Casimirs together with a careful limit procedure have to be performed to derive the VSUSY superspin.

It is interesting to compare this result to the case of ordinary $\mathcal{N} = 1$ SUSY in four dimensions. The corresponding superalgebra can be derived by contraction from OSp(1|4) [12], which has two independent Casimirs. Contraction of the first leads to P^2, while contraction of the second leads in fact to the superspin operator, but in a form that is not at all familiar to physicists (see for instance [13−15]). Therefore, in the case of ordinary $\mathcal{N} = 1$ SUSY, all Casimir operators can be obtained by direct contraction from the OSp embedding algebra.

The paper is organized as follows. In section 2 we introduce the VSUSY algebra and we present its Casimir operators. Moreover, we compare our result to the analogue for ordinary supersymmetry. In section 3 we briefly introduce the OSp$(3,2\,|2)$ algebra and we show how to derive VSUSY by a contraction procedure. In section 4 we discuss the contraction of the Casimir operators and specially how to derive the analogue of superspin for VSUSY. In section 5, we summarize our results and we present our plans for future work. In appendix A, we show in detail how to derive all independent Casimirs of VSUSY. In appendix B, a brief technical introduction to OSp algebras is given and our conventions are stated. Finally, in appendix C, some commutation relations useful for the contraction procedure are given.

2. VSUSY and its Casimir operators

2.1 VSUSY algebra

The vector supersymmetry (VSUSY) algebra in 4 dimensions is a graded algebra defined by the even generators P_μ, $M_{\mu\nu}$, Z, $\tilde{Z}$, and by the odd generators G_μ and G_5. The algebra of the even generators P_μ and $M_{\mu\nu}$ is the usual Poincaré algebra, whereas the odd generators behave respectively as a four-vector and a scalar under the Lorentz group and are translationally invariant. The non-zero (anti)commutation relations are

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}]_- = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma}\,, \\
&[M_{\mu\nu}, P_\rho]_- = -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu\,, \quad [M_{\mu\nu}, G_\rho]_- = -\eta_{\mu\rho} G_\nu + \eta_{\nu\rho} G_\mu\,, \\
&[G_\mu, G_\nu]_+ = \eta_{\mu\nu} Z\,, \qquad [G_5, G_5]_+ = \tilde{Z}\,, \qquad [G_\mu, G_5]_+ = -P_\mu\,,
\end{aligned} \tag{2.1}$$

where we use $[\cdot,\cdot]_-$ for commutators and $[\cdot,\cdot]_+$ for anticommutators. Here and after, the following conventions will be used for the metric and the Levi Civita tensor

$$\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = 1\,. \tag{2.2}$$

Two remarks should be made concerning these equations. First, the VSUSY algebra makes perfect sense as a real algebra. The 'i' factors appearing in [2] were introduced for physical reasons. Since in this paper we are mainly concerned with algebraic properties, we omit them. The complex version of the commutation relations can be obtained by replacing $M_{\mu\nu}$ with $iM_{\mu\nu}$ in (2.1). Second, Z and $\tilde{Z}$ are central charges. Therefore, in any representation they can be considered as numbers.

In the case of non-vanishing Z and $\tilde{Z}$, only their signs and their product are relevant. This can be seen by rescaling

$$\widehat{G}_\mu = \frac{1}{\alpha} G_\mu\,, \qquad \widehat{G}_5 = \alpha G_5\,. \tag{2.3}$$

As a result, the odd sector of the algebra becomes

$$[\widehat{G}_\mu, \widehat{G}_\nu]_+ = \eta_{\mu\nu}\widehat{Z} = \eta_{\mu\nu}\frac{1}{\alpha^2}Z\,, \qquad [\widehat{G}_5, \widehat{G}_5]_+ = \widehat{\tilde{Z}} = \alpha^2 \tilde{Z}\,, \qquad [\widehat{G}_\mu, \widehat{G}_5]_+ = -P_\mu\,. \tag{2.4}$$

We can choose $\alpha^2 = \sqrt{\frac{|Z|}{|\tilde{Z}|}}$, so that the two new central charges have the same absolute value $|\widehat{Z}| = |\widehat{\tilde{Z}}| = \sqrt{|Z\tilde{Z}|} = c$ and the algebra is specified by the value of c and by the signs of Z and $\tilde{Z}$ as

$$[\widehat{G}_\mu, \widehat{G}_\nu]_+ = \eta_{\mu\nu}\mathrm{sign}(Z)\,c\,, \qquad [\widehat{G}_5, \widehat{G}_5]_+ = \mathrm{sign}(\tilde{Z})\,c\,, \qquad [\widehat{G}_\mu, \widehat{G}_5]_+ = -P_\mu\,. \tag{2.5}$$

2.2 Casimir operators

The central charges Z and $\tilde{Z}$ are trivial Casimirs of VSUSY. It is also easy to see that P^2 is a Casimir for VSUSY. As in the case of ordinary supersymmetry, we expect that an analogue of superspin [10] could be constructed by starting from a generalization of the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}\,. \tag{2.6}$$

W^2 itself is not a Casimir. As a result, particles of different Lorentz spin will appear in the same multiplet. The correct VSUSY generalization of the Pauli-Lubanski vector is

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu (ZM_{\rho\sigma} - G_\rho G_\sigma)\,, \tag{2.7}$$

whose square $\hat{W}^2$ is a Casimir. More details concerning how to derive these Casimir operators and how to prove that there are no further independent ones are given in appendix A.

By introducing the new vector

$$W^\mu_C = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu G_\rho G_\sigma\,, \tag{2.8}$$

one can rewrite formula (2.7) as

$$ZW^\mu = \hat{W}^\mu + W_C^\mu \,. \tag{2.9}$$

One can easily prove that W_C^2 is also a Casimir. However, it is not independent, since

$$W_C^2 = Z^2 P^2 \, \frac{3}{4} \,. \tag{2.10}$$

From now on, we are implicitly considering the case of representations with $Z \neq 0$. In that case Z is just a number and can be divided out. The three vectors $W_*^\mu = \frac{\hat{W}^\mu}{Z}$, W^μ, $\frac{W_C^\mu}{Z}$ all commute with P_μ and G_5 and verify the relation

$$[W_*^\mu, W_*^\nu]_- = \epsilon^{\mu\nu\rho\sigma} P_\rho W_{*\sigma} \,. \tag{2.11}$$

Therefore, in the rest frame of the massive states where $P^2 = -m^2$, they satisfy the rotation algebra

$$\left[\frac{W_*^i}{m}, \frac{W_*^j}{m}\right]_- = \epsilon^{ijk} \, \frac{W_{*k}}{m} \,, \tag{2.12}$$

and define three different spins. The superspin Y labels the eigenvalues $-m^2 Z^2 Y(Y+1)$ of the Casimir $\hat{W}^2$. The spin associated to W_C^2 (C-spin) is fixed to 1/2, as one can see from (2.10). Finally, we denote the usual Lorentz spin by s.

On the other hand, only $W_*^\mu = \frac{1}{Z}\hat{W}^\mu$ commutes with G_λ, and thus only the superspin Y characterizes a multiplet. Since

$$\left[\hat{W}^\mu, W_C^\nu\right]_- = 0 \,, \tag{2.13}$$

one can immediately obtain the particle content of a VSUSY multiplet by using the formal theory of addition of angular momenta applied to (2.9). As a result, a multiplet of superspin Y contains two particles of Lorentz spin $Y \pm 1/2$, for $Y > 0$ integer or half-integer. In the degenerate case of superspin $Y = 0$, the multiplet consists of two spin 1/2 states. In particular, we observe that a VSUSY multiplet contains either only particles of half-integer Lorentz spin or only particles of integer Lorentz spin. The spinning particle constructed in [2] is a realization of the degenerate case $Y = 0$.

To summarize, we draw the following table 1. The last column refers to the particle model in [2]. We stress that the eigenvalues appearing in the table are all negative due to the fact that we have chosen a real algebra and antihermitian operators.

2.3 Superspin and Lorentz spin for ordinary supersymmetry

Both for ordinary supersymmetry and VSUSY it is possible to construct a superspin Casimir operator starting from a generalization of the Pauli-Lubanski vector. In this section we would like to revisit the construction of the superspin Casimir for ordinary supersymmetry along the lines of what we have done for VSUSY.

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	eigenvalue			vector superparticle
$\frac{1}{Z^2}\hat{W}^2$	$-m^2\, Y(Y+1)$	superspin $= Y$	Casimir	$Y=0$
$\frac{1}{Z^2}W_C^2$	$-m^2\,\frac{3}{4}$	C spin $= \frac{1}{2}$	Casimir	$C=\frac{1}{2}$
W^2	$-m^2\, s(s+1)$	Lorentz spin $= s = \lvert Y \pm \frac{1}{2}\rvert$	not Casimir	$s=\frac{1}{2}$

Table 1: Properties of three spins and eigenvalues.

We normalize the supersymmetries by assuming that the anticommutator between components Q^α of the spinorial supersymmetry charge Q has the form

$$\left[Q^\alpha, Q^\beta\right]_+ = 2\left(\gamma^\mu C^{-1}\right)^{\alpha\beta}\,, \tag{2.14}$$

where C is the antisymmetric charge conjugation matrix used to define $\bar{Q} = Q^T C$. The suitable generalization of the Pauli-Lubanski vector W^μ reads [10]

$$Z^\mu = W^\mu - \frac{1}{8}\bar{Q}\gamma^\mu\gamma_5 Q\,, \tag{2.15}$$

where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, so that $(\gamma_5)^2 = -1$ and $\gamma^{\mu\nu}\gamma_5 = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}$. Its commutator with the supersymmetry charge is

$$[Z^\mu, Q]_- = \frac{1}{2}P^\mu Q\,, \tag{2.16}$$

such that $Z^\mu P^\nu - Z^\nu P^\mu$ commutes with Q. The superspin Casimir operator is usually written in the literature in the form

$$\mathcal{C} = \frac{1}{2}\left(Z_\mu P_\nu - Z_\nu P_\mu\right)\left(Z^\mu P^\nu - Z^\nu P^\mu\right) = Z^2 P^2 - (Z\cdot P)^2\,. \tag{2.17}$$

In the representations where the Casimir P^2 is nonvanishing, we can equivalently consider $\frac{\mathcal{C}}{P^2}$ as the superspin Casimir. The latter can be expressed as the square of the vector

$$\hat{W}^\mu = Z^\nu\left(\delta_\nu{}^\mu - \frac{P_\nu P^\mu}{P^2}\right) = W^\mu - \frac{1}{8}\bar{Q}\gamma^\nu\gamma_5 Q\left(\delta_\nu{}^\mu - \frac{P_\nu P^\mu}{P^2}\right)\,. \tag{2.18}$$

Then, exactly as in the case of VSUSY, the Pauli-Lubanski vector is the sum of two commuting vectors [11],

$$W^\mu = \hat{W}^\mu + \tilde{W}^\mu_C\,. \tag{2.19}$$

where

$$\tilde{W}^\mu_C = \frac{1}{8}\bar{Q}\gamma^\nu\gamma_5 Q\left(\delta_\nu{}^\mu - \frac{P_\nu P^\mu}{P^2}\right)\,. \tag{2.20}$$

Moreover, as in the VSUSY case, the three vectors $W^\mu_* = W^\mu, \hat{W}^\mu, \tilde{W}^\mu_C$ satisfy

$$[W^\mu_*, W^\nu_*]_- = \epsilon^{\mu\nu\rho\sigma}P_\rho W_{*\sigma}\,. \tag{2.21}$$

Also as in the VSUSY case, $\hat{W}^\mu$ and $\tilde{W}^\nu_C$ commute. However, in this case $\tilde{W}^2_C$ is not a Casimir, in contradistinction to the VSUSY case, where it is proportional to P^2. Despite

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this fact, it is still possible to use this decomposition in the derivation of the particle content of a multiplet for ordinary supersymmetry.

We have

$$\tilde{W}_C^2 = -\frac{3}{32}(\bar{Q}_+ Q_+)(\bar{Q}_- Q_-) + \frac{3}{8}(\bar{Q}_+ \not{P} Q_-)\,. \tag{2.22}$$

where $Q_\pm$ are chiral projections of the Majorana super charge

$$Q_\pm = \mathcal{P}_\pm Q, \qquad \mathcal{P}_\pm = \frac{1}{2}(1 \pm i\gamma_5)\,. \tag{2.23}$$

In terms of these projections, the odd sector of the ordinary supersymmetry algebra can be rewritten as follows

$$\left[Q_+^\alpha, Q_-^\beta\right]_+ = 2(\mathcal{P}_+ \not{P} C^{-1})^{\alpha\beta}\,, \qquad \left[Q_\pm^\alpha, Q_\pm^\beta\right]_+ = 0\,. \tag{2.24}$$

The Hilbert space of the theory contains states of three kinds

$$|Y>, \qquad Q_+^\alpha |Y>, \qquad (\bar{Q}_+ Q_+)|Y>\,, \tag{2.25}$$

with

$$Q_-^\alpha |Y>= 0, \qquad \hat{W}^2 |Y>= -P^2\, Y(Y+1)|Y>\ . \tag{2.26}$$

The values of the C-spin for these states are

$$\begin{aligned} \tilde{W}_C^2 |Y> &= 0\,, \\ \tilde{W}_C^2\, Q_+^\alpha |Y> &= \frac{3}{4} P^2\, Q_+^\alpha |Y>\,, \\ \tilde{W}_C^2 (\bar{Q}_+ Q_+)|Y> &= 0\,. \end{aligned} \tag{2.27}$$

Then $|Y>$ and $(\bar{Q}_+ Q_+)|Y>$ have C-spin 0 and $Q_+^\alpha |Y>$ have C-spin $\frac{1}{2}$. The Lorentz spin is the sum of Y-spin and C-spin. In this way, we rederive the well known result that $|Y>$ and $(\bar{Q}_+ Q_+)|Y>$ have Lorentz spin Y and $Q_+^\alpha |Y>$ have the Lorentz spins $|Y \pm \frac{1}{2}|$. Note the difference with the VSUSY case where the C-spin is fixed to 1/2 and therefore formula (2.9) allows for a direct derivation of the particle content of a multiplet. In the ordinary supersymmetry case, together with the analogous formula (2.19), one needs to know also the Hilbert space structure given by formula (2.25).

2.4 Odd 'Casimir'

For ordinary supersymmetry, there can be no odd Casimirs, as the fermions are spinors and hence do not commute with the Lorentz generators. However, for VSUSY this argument does not hold. We can find an operator that commutes with all generators of the algebra under one condition on the mass and central charges, a BPS-like condition. This is an invariant operator in certain representations of the algebra (we call it 'Casimir' in this paper).

We consider the simplest possibility of an odd operator linear in the anticommuting generators (G_μ, G_5). The set of conditions one obtains when imposing that such operator commutes with G_μ and G_5 admit a nontrivial solution when the determinant of the

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following matrix is zero:

$$[G_A, G_B]_+ = \begin{bmatrix} \eta_{\mu\nu} Z & -P_\mu \\ -P_\nu & \tilde{Z} \end{bmatrix}, \qquad G_A = (G_\mu, G_5)\,. \tag{2.28}$$

The determinant vanishes when the following BPS-like condition is satisfied:

$$Z\tilde{Z} + m^2 = 0\,. \tag{2.29}$$

In this case, the matrix (2.28) admits the eigenvector (P_ν, Z) with zero eigenvalue. Therefore, the odd 'Casimir' we are looking for has the form

$$Q = G \cdot P + G_5 Z\,. \tag{2.30}$$

It commutes with all the generators under the condition (2.29), implying that when $m \neq 0$ both Z and $\tilde{Z}$ are nonvanishing and have opposite sign. We can rewrite this condition in terms of the variable c introduced at the end of section 2.1 and the signs of the central charges as

$$c = \sqrt{|Z\tilde{Z}|} = |m|\,, \qquad \mathrm{sign}(Z) = -\,\mathrm{sign}(\tilde{Z})\,. \tag{2.31}$$

We notice that Q is of course a 'Casimir' also when $m = 0$, but in that case one or both central charges must be zero. We are not studying this case in this paper.

Q acts as an odd constant on the states in the representations satisfying condition (2.29). Therefore, unless the model under consideration has a natural odd constant, Q has to annihilate all states in those representations. As a result, the physical role of the odd 'Casimir' is to give a Dirac-type equation for the particle states.

In principle, one could also look for odd Casimirs that are cubic or of higher order in the odd generators. In fact, one can prove that such higher order 'Casimirs' do not arise. A brief discussion of this point is presented at the end of appendix A.

3. VSUSY as a contraction of OSp(3, 2 |2)

It is interesting to explore the connection of VSUSY with other algebras. Concerning this point, we have already mentioned in the introduction that the Euclidean version of VSUSY is related to the symmetries of topological $\mathcal{N} = 2$ Yang-Mills theories.

In this section we study the relation of Minkowskian VSUSY with the simple orthosymplectic algebra OSp(3, 2 |2). We show that our algebra arises as a subalgebra of an Inönü-Wigner contraction (further we just write 'contraction') of OSp(3, 2 |2). In fact, the ordinary supersymmetry algebra can also be derived by a similar contraction procedure from OSp(1|4), as shown in [12].

Both ordinary supersymmetry and VSUSY are generalizations of the Poincaré algebra. The Poincaré algebra itself is not a simple algebra. There are, however, two well-known connections to simple algebras. First, the Poincaré algebra can be seen as a contraction of a simple algebra. The simple algebra is then the de Sitter algebra, where one introduces a scale λ, such that for $\lambda \to \infty$ the Poincaré algebra results using the same generators. Another procedure starts from the conformal algebra, of which Poincaré is a subalgebra.

For superalgebras the same facts hold. The ordinary supersymmetry algebra is a contraction of a super-de Sitter algebra and a subalgebra of a superconformal algebra, which are both simple superalgebras. Apart from a few exceptions, the infinite series of superalgebras is either a generalization of $\mathrm{U}(N)$, i.e. $\mathrm{SU}(N|M)$ superalgebras, or a generalization of $\mathrm{SO}(N)$, i.e. $\mathrm{OSp}(N|M)$, which can also be seen as generalizations of $\mathrm{Sp}(M)$.

A tricky point for ordinary supersymmetry is that the fermions should be in spin representations, while in $\mathrm{OSp}(N|M)$ the fermions are vectors of $\mathrm{SO}(N)$ and of $\mathrm{Sp}(M)$. Therefore, the bosonic spacetime group in these superalgebras can not be recognized as the $\mathrm{SO}(N)$ subalgebra of $\mathrm{OSp}(N|M)$ (with N including both signatures). Instead, we have to use e.g. for 4 dimensions the equivalences $\mathrm{Spin}(3,2) = \mathrm{Sp}(4)$ for the (anti)-de Sitter algebra and $\mathrm{Spin}(4,2) = \mathrm{SU}(2,2)$ for the conformal algebra. Then the superalgebras that can be used are respectively $\mathrm{OSp}(N|4)$ and $\mathrm{SU}(2,2|N)$ [16], so that the fermions, being vectors of $\mathrm{Sp}(4)$ or $\mathrm{SU}(2,2)$ are spinors of $\mathrm{SO}(3,2)$ or $\mathrm{SO}(4,2)$. For VSUSY we do not have this difficulty. We want the fermions to appear as a vector (or a scalar) of the Lorentz group. Therefore we will use the embedding in $\mathrm{OSp}(3,2\,|2)$, whose bosonic part is $\mathrm{SO}(3,2) \times \mathrm{Sp}(2)$.

A brief introduction to OSp algebras including more technical details is given in appendix B. For what follows, it is enough to know that the $\mathrm{OSp}(3,2\,|2)$ generators are $M_{\mu\nu}$, P_μ, Z, $\tilde{Z}$, Z' (bosonic) and G_μ, S_μ, G_5, S_5 (fermionic). The subset of generators $(M_{\mu\nu}, P_\mu, Z, \tilde{Z}, G_\mu, G_5)$ has the correct structure to generate the VSUSY algebra. However, the $\mathrm{OSp}(3,2\,|2)$ commutation relations for the sector of interest are[2]

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}]_- = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}\,,\\
&[M_{\mu\nu}, P_\rho]_- = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu\,,\\
&[P_\mu, P_\nu]_- = M_{\mu\nu}\,,\\
&[M_{\mu\nu}, G_\rho]_- = \eta_{\nu\rho} G_\mu - \eta_{\mu\rho} G_\nu\,,\\
&[P_\mu, G_\nu]_- = -\eta_{\mu\nu} S_5\,, \qquad [P_\mu, G_5]_- = -S_\mu\,, \qquad [G_\mu, G_\nu]_+ = \eta_{\mu\nu} Z\,,\\
&[G_\mu, G_5]_+ = -P_\mu\,, \qquad [G_5, G_5]_+ = \tilde{Z}\,,\\
&\left[G_\mu, \tilde{Z}\right]_- = 2S_\mu\,, \qquad [G_5, Z]_- = 2S_5\,, \qquad \left[Z, \tilde{Z}\right]_- = 4Z'\,.
\end{aligned}
\tag{3.1}
$$

Therefore, it is clear that VSUSY is not a subalgebra of $\mathrm{OSp}(3,2\,|2)$, but it could arise after a proper contraction. In order to do this, we rescale the $\mathrm{OSp}(3,2\,|2)$ generators with a dimensionless parameter λ as follows:

$$
\begin{aligned}
&M_{\mu\nu} \to M_{\mu\nu}\,, \qquad Z' \to Z'\,,\\
&P_\mu \to \lambda^2 P_\mu\,, \qquad Z \to \lambda^2 Z\,, \qquad \tilde{Z} \to \lambda^2 \tilde{Z}\,,\\
&G_\mu \to \lambda G_\mu\,, \qquad G_5 \to \lambda G_5\,, \qquad S_\mu \to \lambda S_\mu\,, \qquad S_5 \to \lambda S_5\,,
\end{aligned}
\tag{3.2}
$$

and consider the limit $\lambda \to \infty$. As a result, the commutation relations (3.1) reduce to the VSUSY algebra (2.1).[3]

[2] For completeness, the remaining commutation relations are given in (B.7) in appendix B.

[3] We give the contraction limit of the remaining commutation relations of $\mathrm{OSp}(3,2\,|2)$ in appendix C.

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Therefore, we can conclude that the VSUSY algebra is a subalgebra of the contraction limit of OSp(3, 2 |2).

4. Contraction of the Casimir operators

In this section we derive the Casimir operators of VSUSY by contraction from OSp(3, 2 |2).

OSp(3, 2 |2) has 3 independent Casimir operators [17], $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_6$, of the form

$$\mathcal{C}_n = str(M^n) = \sum(-1)^A M^A{}_B M^B{}_C \dots M^C{}_D M^D{}_A \,, \tag{4.1}$$

where the generators M_{AB} of OSp(3, 2 |2) are parametrized as in (B.6) in appendix B. Explicitly, the quadratic Casimir reads

$$\mathcal{C}_2 = M_{\mu\nu}M^{\nu\mu} + 2P_\mu P^\mu + 2\,[G_\mu, S^\mu]_- + 2\,[G_5, S_5]_- - 2Z'^2 - [Z, \tilde{Z}]_+ \,. \tag{4.2}$$

We do not give explicitly the lengthy formulas for $\mathcal{C}_4$ and $\mathcal{C}_6$. In the following, we have used Mathematica-coding based on the superEDC package developed by Bonanos [18] to derive and handle such expressions.

We have shown in the previous section that VSUSY is only a subalgebra of the contraction limit of OSp(3, 2 |2). Therefore, when we take the contraction limit of the Casimir operators of OSp(3, 2 |2), we obtain Casimir operators of the contracted algebra, containing VSUSY as a subalgebra. In order to obtain Casimir operators of VSUSY, we have to eliminate the extra operators S_μ, S_5 and Z'. For our purposes, the best way to do this is to introduce a second parameter β and make a rescaling

$$S_\mu \to \beta S_\mu \,, \qquad S_5 \to \beta S_5 \,, \qquad Z' \to \beta Z' \,. \tag{4.3}$$

By taking the limit $\beta \to 0$ we reduce to the VSUSY subsector.

By inspection of the λ scaling relations (3.2), it is clear that the combinations P^2 and $Z\tilde{Z}$ scale with maximal power in λ, so that the direct contraction of the three OSp(3, 2 |2) Casimirs $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_6$ can only lead to combinations of these quantities and explicitly one finds

$$\mathcal{C}_n \to 2\left((P^2)^{\frac{n}{2}} - (Z\tilde{Z})^{\frac{n}{2}}\right) \,, \qquad n = 2, 4, 6 \,, \tag{4.4}$$

which are clearly Casimirs of VSUSY. Since Z and $\tilde{Z}$ are central charges, the new information is that P^2 is a Casimir of VSUSY. Therefore, we can not obtain the VSUSY superspin Casimir $\hat{W}^2$ (2.7) from a direct contraction procedure, since $M_{\mu\nu}$ does not scale in λ, see (3.2), and the corresponding term in the superspin Casimir does not have maximal order in λ.

A way out is to start from a homogeneous polynomial in the OSp(3, 2 |2) Casimirs $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_6$ of suitable degree, characterized by the fact that the terms of maximal order in λ exactly cancel out.

Since $\hat{W}^2$ is of order 6 in the VSUSY generators, the simplest possibility would be to start from a polynomial of order 6. However, inspection of the general structure of the OSp(3, 2 |2) Casimirs shows that a term with two Z's, two P^μ's and two $M_{\mu\nu}$'s will never appear, due to the symmetric behavior with respect to Z and $\tilde{Z}$.

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Therefore, we move on to the next nontrivial order by considering a polynomial of order 8. One can then show that, by imposing the vanishing of the maximal order terms in λ (λ^{16}), the following combination is selected uniquely:

$$\mathcal{K}_8 = -4\,\mathcal{C}_6\,\mathcal{C}_2 + 3\,\mathcal{C}_4^2 + \frac{1}{4}\,\mathcal{C}_2^4\,. \tag{4.5}$$

For the contraction limit to give the leading terms of order λ^{12} as a result, it is necessary that not only the λ^{16}, but also the intermediate orders λ^{15}, λ^{14} and λ^{13} cancel out. In fact, the terms of odd power in λ are absent, due to the fact that $\mathcal{K}_8$ is even and only fermionic generators scale with odd powers of λ. We have then checked explicitly that the term of order λ^{14} can be reduced to terms of order 12 or lower. To obtain this result, we have used the (rescaled) commutation relations in (C.1)–(C.3), given in appendix C, and we have therefore produced some extra terms of lower order in λ. The order 12 terms are the interesting ones for the derivation of the superspin Casimir because this is the first place where $M_{\mu\nu}$ terms appear. We have checked that the extra terms at order 12 generated from the higher order terms by the use of commutation relations vanish after the limit $\lambda \to \infty$ is taken. This makes sure that they will not affect the result of the contraction. We thus have

$$\mathcal{K}_8 = \lambda^{12}\mathcal{K}_8^{(12)} + \lambda^{10}\mathcal{K}_8^{(10)} + \dots\,. \tag{4.6}$$

Up to now we have proven that the contraction limit $\lambda \to \infty$ of $\mathcal{K}_8$ gives the order λ^{12} term as a result. We still have to show how this term is connected to the superspin Casimir $\hat{W}^2$.

Indeed, one can prove the following relation

$$\mathcal{K}_8^{(12)} = 48\left(P^2 - Z\tilde{Z}\right)\left(-\frac{\tilde{Z}}{Z}\hat{W}^2 + \beta^2\{\text{terms with}(S_\mu, S_5, Z')\}\right) + f(P^2, Z, \tilde{Z})\,. \tag{4.7}$$

This analysis proves that $\hat{W}^2$ is a Casimir operator of the VSUSY algebra as a consequence of the fact that $\mathcal{K}_8$ is a Casimir of $\mathrm{OSp}(3,2\,|2)$. Indeed, this includes the statement that any generator T in the VSUSY algebra commutes with $\mathcal{K}_8$ and hence

$$0 = \lambda^{-12}\Big[T, \mathcal{K}_8\Big]_- = \Big[T, \mathcal{K}_8^{(12)}\Big]_- + \mathcal{O}(\lambda^{-1})\,. \tag{4.8}$$

We keep in principle the λ and β-dependent commutators in (C.1)–(C.3), and the second equality holds because these do not involve positive powers of λ. Then we use (4.7) and the fact that P^2, Z and $\tilde{Z}$ were already recognized as Casimirs of the VSUSY algebra (i.e. they commute with T up to λ^{-1} terms) and in the leading order of λ commutators with T produce according to (C.2) at most terms of order β^{-1}. This leads to

$$0 = 48\left(P^2 - Z\tilde{Z}\right)\left(-\frac{\tilde{Z}}{Z}\Big[T, \hat{W}^2\Big]_- + \beta^2 b_2 + \beta b_1\right) + \mathcal{O}(\lambda^{-1})\,, \tag{4.9}$$

where b_2 and b_1 are functions of the operators whose explicit form is not important. Therefore, if we take first the limit $\lambda \to \infty$ and then the decoupling limit $\beta \to 0$, we obtain that using the commutators of the VSUSY algebra (i.e. dropping λ^{-1} terms)

$$\Big[T, \hat{W}^2\Big]_- = 0\,. \tag{4.10}$$

5. Conclusions and outlook

The aim of this paper is to study the basic algebraic properties of the VSUSY algebra and its connections with other algebras. Our results will hopefully shed light on the classification of the irreducible representations of the algebra, or, at least, will help in the identification of a class of physically interesting ones. The representations of VSUSY are not discussed in this paper. We leave this for future work.

VSUSY shares some common features with ordinary supersymmetry, for instance the fact that the anticommutator between the fermionic generators is proportional to the four-momentum P_μ. On the other hand, the fundamental difference between supersymmetry and VSUSY is the Lorentz nature of their odd generators, spinors for supersymmetry and a vector and a scalar for VSUSY. We found that in the case $Z, \tilde{Z} \neq 0$ VSUSY has four independent even Casimir operators, $\hat{W}^2$, P^2, Z and $\tilde{Z}$. We have also been able to construct an odd operator Q, which is nilpotent and behaves like a Casimir when a BPS-like relation between the central charges and the four-momentum is satisfied ($Z\tilde{Z} = P^2$).

The Casimir operator $\hat{W}^2$ is the square of a Lorentz vector $\hat{W}_\mu$, which is the VSUSY extension of the ordinary Pauli-Lubanski vector. In the rest frame, it satisfies the SU(2) algebra and gives rise to the superspin Y, the analogue of superspin for VSUSY. We want to stress that it is necessary to have both central charges different from zero to ensure that this superspin operator is an independent Casimir. In fact, in the case $Z = \tilde{Z} = 0$ it collapses to P^2, up to a constant. On the other hand, the Casimir operator P^2 is related to another Lorentz vector, denoted by W_C^μ. In the rest frame, W_C^μ also satisfies an SU(2) algebra and defines a different kind of spin, fixed to the value $\frac{1}{2}$. As a result of the algebraic relation among the three spin-generating vectors, a multiplet consists of a doublet of spin $(s, s+1)$ or two spin 1/2 states.

In this paper we have also investigated the relations between VSUSY and other algebras. First of all, we have observed that an Euclidean version of VSUSY is a subalgebra of the $\mathcal{N} = 2$ topological algebra. Furthermore, by exploiting the fact that VSUSY displays fermionic generators which are a vector and a scalar, we have shown how the (Minkowskian) VSUSY generators are naturally embedded in the simple orthosymplectic superalgebra $\mathrm{OSp}(3, 2\,|2)$.

We have derived the VSUSY algebra and all its independent Casimirs from $\mathrm{OSp}(3, 2\,|2)$ by performing a suitable contraction limit.

The issue of classification of VSUSY irreducible representations remains. One possibility in this direction is to rewrite the odd sector of the VSUSY algebra in terms of the generators of a five-dimensional or six-dimensional Clifford algebra, for which all the irreducible representations have already been classified. Another possibility would be to exploit the embedding of the VSUSY algebra in $\mathrm{OSp}(3, 2\,|2)$ to derive the representations. One of the final goals would be to construct physically relevant models with underlying VSUSY. A first example of a particle model is given in [2]. In a field theory context, it would be nice to develop a superspace formalism for VSUSY. In this direction, the connection between VSUSY and $\mathcal{N} = 2$ topological theories could turn out to be useful, since a (twisted) superspace setup has already been constructed in that case [6]. We are currently working on these developments [19].

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Acknowledgments

We thank Dan Freedman, Gary Gibbons, Satoshi Iso, Yoshihisa Kitazawa and Tomás Ortín for useful discussions. This work has been partially supported by MCYT FPA 2007-66665, CIRIT GC 2005SGR-00564, Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042), FWO - Vlaanderen, project G.0235.05 and the Federal Office for Scientific, Technical and Cultural Affairs through the 'Interuniversity Attraction Poles Programme – Belgian Science Policy' P6/11-P.

A. Casimirs of VSUSY

In this appendix we present a general procedure to derive all independent Casimir operators of VSUSY.

We start from the most general form of an even Casimir operator of the VSUSY algebra, which reads

$$\mathcal{C} = C + C^{\mu\nu} G_\mu G_\nu + C^{\mu 5} G_\mu G_5 + C^* \epsilon^{\mu\nu\rho\sigma} G_\mu G_\nu G_\rho G_\sigma + C^*_\mu \epsilon^{\mu\nu\rho\sigma} G_\nu G_\rho G_\sigma G_5 \,, \quad \text{(A.1)}$$

where the coefficients C's are functions of the bosonic generators $(P, M, Z, \tilde{Z})$ and $C^{\mu\nu}$ is antisymmetric. Any even products of G's can be arranged in the above form using the algebra (2.1).

The condition $[\mathcal{C}, G_5]_- = 0$ implies

$$2C^{\mu\nu} P_\nu - C^{\mu 5} \tilde{Z} = 0 \,, \qquad C^{\mu 5} P_\mu = 0 \,, \qquad 4C^* P_\mu + C^*_\mu \tilde{Z} = 0 \,, \qquad C^*_\mu \epsilon^{\mu\nu\rho\sigma} P_\sigma = 0 \,, \quad \text{(A.2)}$$

and for non zero $\tilde{Z}$ we solve these for $C^{\mu 5}$ and C^*_μ and obtain

$$\mathcal{C} \;=\; C + C^{\mu\nu} \tilde{G}_\mu \tilde{G}_\nu + C^* \epsilon^{\mu\nu\rho\sigma} \tilde{G}_\mu \tilde{G}_\nu \tilde{G}_\rho \tilde{G}_\sigma \,. \quad \text{(A.3)}$$

Here $\tilde{G}_\mu$ is defined by

$$\tilde{G}_\mu \equiv G_\mu + \frac{1}{\tilde{Z}} P_\mu G_5 \,, \quad \text{(A.4)}$$

and satisfies

$$[\tilde{G}_\mu, G_5]_+ = 0 \quad \text{and} \quad [\tilde{G}_\mu, \tilde{G}_\nu]_+ = \eta_{\mu\nu} Z - \frac{P_\mu P_\nu}{\tilde{Z}} \,. \quad \text{(A.5)}$$

The invariance of $\mathcal{C}$ with respect to the Poincaré subgroup implies that the C's transform as Lorentz covariant tensors and that they are functions of $(P_\mu, W_\mu, Z, \tilde{Z})$. The covariance requires that C, $C_{\mu\nu}$ and C^* have the following form

$$\begin{aligned} C \;&=\; C(P^2, W^2, Z, \tilde{Z}) \,, \qquad\qquad C^* = C^*(P^2, W^2, Z, \tilde{Z}) \,, \\ C^{\mu\nu} \;&=\; C'(P^2, W^2, Z, \tilde{Z}) \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma + C''(P^2, W^2, Z, \tilde{Z}) P^{[\mu} W^{\nu]} \,. \end{aligned} \quad \text{(A.6)}$$

The remaining condition $[\mathcal{C}, G_\mu]_- = 0$, or, equivalently, $[\mathcal{C}, \tilde{G}_\mu]_- = 0$ has then to be considered. We have three Casimirs independent of $\tilde{G}$'s,

$$P^2 \,, \quad Z \quad \text{and} \quad \tilde{Z} \,, \quad \text{(A.7)}$$

whereas W^2 is not a Casimir of the VSUSY algebra.

The most general structure of a Casimir of second order in $\tilde{G}$ is

$$\mathcal{C}_{(2)} = C(P^2, W^2, Z, \tilde{Z}) + C' \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma \tilde{G}_\mu \tilde{G}_\nu + C''(P^\mu W^\nu - P^\nu W^\mu)\tilde{G}_\mu \tilde{G}_\nu \,. \tag{A.8}$$

We start by considering the commutator of the second term with $\tilde{G}_\lambda$. After some algebraic manipulations we obtain

$$[\epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma \tilde{G}_\mu \tilde{G}_\nu, \tilde{G}_\lambda]_- = 2Z\left(\epsilon^{\mu\nu\rho}{}_\lambda P_\mu W_\nu \tilde{G}_\rho + (P^2 \tilde{G}_\lambda - (P\tilde{G})P_\lambda)\right) = -Z[W^2, \tilde{G}_\lambda]_- \,. \tag{A.9}$$

This equation can be written as

$$[ZW^2 + \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma \tilde{G}_\mu \tilde{G}_\nu, \tilde{G}_\lambda]_- = 0 \,. \tag{A.10}$$

Therefore, we have found a Casimir with a second order term in $\tilde{G}_\mu$:

$$\mathcal{C}_{(2)} = ZW^2 + \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma \tilde{G}_\mu \tilde{G}_\nu = ZW^2 + \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma G_\mu G_\nu \,. \tag{A.11}$$

In the previous formula, the P_μ terms in $\tilde{G}_\mu$ do not contribute. It is convenient to introduce a vector that is a polynomial in the generators,

$$\hat{W}^\mu \equiv ZW^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu G_\rho G_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu (ZM_{\rho\sigma} - G_\rho G_\sigma) \,. \tag{A.12}$$

Its square gives an alternative form of the Casimir operator $\mathcal{C}_{(2)}$ in (A.11) for the VSUSY algebra since

$$\hat{W}^2 = Z\mathcal{C}_{(2)} + \frac{3}{4}P^2 Z^2 \,. \tag{A.13}$$

The commutator of the C'' term in (A.8) with $\tilde{G}_\tau$ has a linear and a cubic part in $\tilde{G}_\mu$. In principle, these terms could cancel with the contributions coming from the quartic term in (A.3). We obtain with a convenient normalization ($C'' = 1/2$ and $C^* = \frac{b}{12}\tilde{Z}$)

$$\begin{aligned}
&\left[P^{[\mu}W^{\nu]}\tilde{G}_\mu \tilde{G}_\nu + \frac{b}{12}\tilde{Z}\epsilon^{\mu\nu\rho\sigma}\tilde{G}_\mu \tilde{G}_\nu \tilde{G}_\rho \tilde{G}_\sigma, \tilde{G}_\tau\right]_- \\
&\qquad = ZP^\mu \tilde{G}_\mu W_\tau - \frac{1}{\tilde{Z}}\left(Z\tilde{Z} - P^2\right) W^\mu \tilde{G}_\mu P_\tau \\
&\qquad\quad + \frac{1}{3}\epsilon^{\mu\nu\rho\sigma}\tilde{G}_\mu \tilde{G}_\nu \tilde{G}_\rho \left((1-b)P_\sigma P_\tau + (bZ\tilde{Z} - P^2)\eta_{\sigma\tau}\right) \,.
\end{aligned} \tag{A.14}$$

It is tantalizing that all but the first term cancels for $b = 1$ when the BPS-like condition $Z\tilde{Z} - P^2 = 0$ holds. However, the first term remains and so the hoped cancellation does not occur.

Finally, one may look for higher order odd 'Casimirs'. For $\tilde{Z} \neq 0$, as we always assume in this paper, in the algebra written in terms of $\tilde{G}_\mu$ and G_5, the only nontrivial commutator involving G_5 is the one between G_5 and itself. Therefore, requiring that such a Casimir commutes with G_5 implies that G_5 can not explicitly appear, and the only expression that we should look at is

$$Q^{(3)} = B^\mu \tilde{G}_\mu + A_\mu \epsilon^{\mu\nu\rho\sigma}\tilde{G}_\nu \tilde{G}_\rho \tilde{G}_\sigma \,, \tag{A.15}$$

where A_μ and B_μ are bosonic vectors, functions of P_μ, W_μ, Z and $\tilde{Z}$. We impose that this commutes with $\widetilde{G}_\lambda$, under the condition (2.29). It can be easily checked that only $B^\mu = P^\mu$ and $A_\mu = 0$ give a solution, which is the one mentioned in (2.30).

In summary, we have shown that there are four even Casimir operators of the VSUSY algebra,

$$Z\,, \qquad \tilde{Z}\,, \qquad P^2 \quad \text{and} \quad \hat{W}^2\,, \tag{A.16}$$

and the odd 'Casimir' Q of (2.30) in representations satisfying (2.29),

$$Q = G\cdot P + G_5 Z\,, \qquad P^2 - Z\tilde{Z} = 0\,. \tag{A.17}$$

B. Definition and conventions for OSp algebras

OSp algebras are very simple generalizations of SO or Sp algebras. The SO algebras have a symmetric metric, the Sp algebras have an antisymmetric metric, and the OSp algebras have a 'graded symmetric' metric.

To understand graded symmetry, one needs the supertranspose of tensors. If one has graded indices $A, B, \ldots$ which are either bosonic (then $(-)^A = 1$) or fermionic (with $(-)^A = -1$), supertranspose acts differently according to whether an index is upper or lower. We have

$$\begin{aligned}
T^{AB} &: \text{supertranspose}: \ (-)^{AB}T^{BA}\,,\\
T_{AB} &: \text{supertranspose}: \ (-)^{AB+A+B}T_{BA}\,,\\
T^A{}_B &: \text{supertranspose}: \ (-)^{AB+B}T_B{}^A\,,\\
T_A{}^B &: \text{supertranspose}: \ (-)^{AB+A}T^B{}_A\,,
\end{aligned} \tag{B.1}$$

i.e., apart from the $(-)^{AB}$ factor, there is an extra factor when a lower index changes from first to last position. A supertrace is made with a factor $(-)^A$:

$$str\,T = (-)^A T_A{}^A\,, \qquad \text{or} \qquad str\,T = (-)^A T^A{}_A\,, \tag{B.2}$$

which means that this definition is invariant under supertranspose. Moreover, the supertranspose of the product of matrices $M\,N$ is $N^T\,M^T$.

A general treatment is given in [20]. The matrices that we use are all of 'bosonic type' in the terminology of this book.

The superalgebra OSp consists of matrices preserving a graded symmetric metric η_{AB}. When we use α for the part of the indices A that are bosonic and i for those that are fermionic, we can block-diagonalize such that $\eta_{\alpha i} = \eta_{i\alpha} = 0$, $\eta_{\alpha\beta} = \eta_{\beta\alpha}$ and $\eta_{ij} = -\eta_{ji}$. We use $\eta_{\alpha\beta} = \text{diag}(-1,+1,+1,+1,-1)$ and $\eta_{21} = 1$ for the $\text{OSp}(3,2\,|2)$ metric. The generators M_{AB} are graded antisymmetric, which thus means that

$$M_{AB} = (-)^{(A+1)(B+1)}M_{BA}\,, \tag{B.3}$$

and we can use the inverse of η to raise and lower indices, having the care of putting summed indices always in adjacent positions.

In order to obtain the commutation relations with correct signs we should use the structures defined above. A convenient way consists in forming a supertrace of the generators and parameters λ^{AB} which are graded antisymmetric. This leads to

$$[M_{AB}, M_{CD}]_{\pm}\, \lambda^{DC}(-)^{C} = 2M_{AD}\lambda^{DC}\eta_{CB} + 2(-)^{(A+1)(B+1)} M_{BD}\lambda^{DC}\eta_{CA}\,. \tag{B.4}$$

The left-hand side uses anticommutators or commutators, according to the type of the generators. The right-hand side involves a matrix product of $M\lambda\eta$ and uses the graded symmetry for the transpose and a convenient normalization such that the bosonic subalgebra has the usual normalization for orthogonal algebras.

When we extract the parameters from (B.4), we have to respect another graded antisymmetry and obtain

$$\begin{aligned}[M_{AB}, M_{CD}]_{\pm} = {} & M_{AD}\eta_{CB}(-)^{C} + (-)^{(A+1)(B+1)+C} M_{BD}\eta_{CA} \\ & + M_{AC}\eta_{DB}(-)^{CD+C+1} + (-)^{(AB+A+B+CD+C)} M_{BC}\eta_{DA}\,. \end{aligned} \tag{B.5}$$

In the special case of $\mathrm{OSp}(3,2\,|2)$ considered in this paper, the generators can be organized in a graded anti-symmetric supermatrix as

$$M_{AB} = \begin{bmatrix} M_{\mu\nu} & P_{\mu} & G_{\mu} & S_{\mu} \\ -P_{\nu} & 0 & S_5 & G_5 \\ G_{\nu} & S_5 & Z & Z' \\ S_{\nu} & G_5 & Z' & -\tilde{Z} \end{bmatrix}. \tag{B.6}$$

In this paper, the $\mathrm{OSp}(3,2\,|2)$ commutation relations rewritten in terms of the entries of this matrix are used. They are (3.1) and the other non-vanishing ones are

$$\begin{aligned}
[M_{\mu\nu}, S_{\rho}]_{-} &= \eta_{\nu\rho}S_{\mu} - \eta_{\mu\rho}S_{\nu}\,, \\
[P_{\mu}, S_{\nu}]_{-} &= -\eta_{\mu\nu}G_5\,, \\
[P_{\mu}, S_5]_{-} &= -G_{\mu}\,, \\
[S_{\mu}, S_{\nu}]_{+} &= -\eta_{\mu\nu}\tilde{Z}\,, \\
[G_{\mu}, S_{\nu}]_{+} &= \eta_{\mu\nu}Z' - M_{\mu\nu}\,, \\
[S_{\mu}, S_5]_{+} &= P_{\mu}\,, \\
[S_5, S_5]_{+} &= -Z\,, \\
[S_5, G_5]_{+} &= -Z'\,, \\
[G_{\mu}, Z']_{-} &= -G_{\mu}\,, \\
[S_{\mu}, Z]_{-} &= 2G_{\mu}\,, \qquad [S_{\mu}, Z']_{-} = S_{\mu}\,, \\
\left[S_5, \tilde{Z}\right]_{-} &= 2G_5\,, \qquad [S_5, Z']_{-} = -S_5\,, \\
[G_5, Z']_{-} &= G_5\,, \\
[Z, Z']_{-} &= -2Z\,, \qquad \left[\tilde{Z}, Z'\right]_{-} = 2\tilde{Z}\,.
\end{aligned} \tag{B.7}$$

C. Some useful formulas for the $\mathrm{OSp}(3,2\,|2)$ contraction

In order to perform the contraction of the $\mathrm{OSp}(3,2\,|2)$ algebra only the λ rescaling (3.2) is necessary. However, in order to obtain the Casimirs of VSUSY, as explained in the

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text, we need to perform also a β rescaling (4.3). Therefore, we give in the following the commutation relations rescaled both in λ and β.

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}]_- = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \,, \\
&[M_{\mu\nu}, P_\rho]_- = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \quad [M_{\mu\nu}, G_\rho]_- = \eta_{\nu\rho} G_\mu - \eta_{\mu\rho} G_\nu \,, \\
&[P_\mu, P_\nu]_- = \frac{1}{\lambda^4} M_{\mu\nu} \,, \qquad [P_\mu, G_\nu]_- = -\frac{\beta}{\lambda^2} \eta_{\mu\nu} S_5 \,, \qquad [P_\mu, G_5]_- = -\frac{\beta}{\lambda^2} S_\mu \,, \\
&[G_\mu, G_\nu]_+ = \eta_{\mu\nu} Z \,, \qquad [G_\mu, G_5]_+ = -P_\mu \,, \qquad [G_5, G_5]_+ = \tilde{Z} \,, \\
&\left[G_\mu, \tilde{Z}\right]_- = \frac{\beta}{\lambda^2} 2 S_\mu \,, \qquad [G_5, Z]_- = \frac{\beta}{\lambda^2} 2 S_5 \,, \qquad \left[Z, \tilde{Z}\right]_- = \frac{\beta}{\lambda^4} 4 Z' \,.
\end{aligned}
\tag{C.1}
$$

It is then clear that, by taking the limit $\lambda \to \infty$ with β fixed, this sector contracts to VSUSY, given by (2.1), which is thus a subalgebra of this λ-contracted OSp$(3,2\,|2)$.

The nonzero rescaled commutation relations between the operators S_μ, S_5 and Z' and those of the VSUSY algebra are:

$$
\begin{aligned}
&[M_{\mu\nu}, S_\rho]_- = \eta_{\nu\rho} S_\mu - \eta_{\mu\rho} S_\nu \,, \\
&[P_\mu, S_\nu]_- = -\frac{1}{\beta\lambda^2} \eta_{\mu\nu} G_5 \,, \qquad [P_\mu, S_5]_- = -\frac{1}{\beta\lambda^2} G_\mu \,, \\
&[G_\mu, S_\nu]_+ = \frac{1}{\lambda^2} \eta_{\mu\nu} Z' - \frac{1}{\beta\lambda^2} M_{\mu\nu} \,, \qquad [G_\mu, Z']_- = -\frac{1}{\beta} G_\mu \,, \\
&[G_5, S_5]_+ = -\frac{1}{\lambda^2} Z' \,, \qquad [G_5, Z']_- = \frac{1}{\beta} G_5 \,, \\
&[Z, S_\mu]_- = -\frac{1}{\beta\lambda^2} 2 G_\mu \,, \qquad [Z, Z']_- = -\frac{1}{\beta} 2 Z \,, \\
&\left[\tilde{Z}, S_5\right]_- = -\frac{1}{\beta\lambda^2} 2 G_5 \,, \qquad \left[\tilde{Z}, Z'\right]_- = \frac{1}{\beta} 2 \tilde{Z} \,.
\end{aligned}
\tag{C.2}
$$

The nonzero commutation relations among these extra generators are

$$
\begin{aligned}
&[S_\mu, S_\nu]_+ = -\frac{1}{\beta^2} \eta_{\mu\nu} \tilde{Z} \,, \qquad [S_\mu, S_5]_+ = \frac{1}{\beta^2} P_\mu \,, \qquad [S_5, S_5]_+ = -\frac{1}{\beta^2} Z \,, \\
&[S_\mu, Z']_- = \frac{1}{\beta} S_\mu \,, \qquad [S_5, Z']_- = -\frac{1}{\beta} S_5 \,.
\end{aligned}
\tag{C.3}
$$

The contracted algebra of the OSp$(3,2\,|2)$ is obtained by taking the limit $\lambda \to \infty$ with $\beta = 1$. Apart from the VSUSY algebra (2.1) the nonvanishing commutation relations are

$$
\begin{aligned}
&[M_{\mu\nu}, S_\rho]_- = \eta_{\nu\rho} S_\mu - \eta_{\mu\rho} S_\nu \,, \\
&[S_\mu, S_\nu]_+ = -\eta_{\mu\nu} \tilde{Z} \,, \qquad [S_\mu, S_5]_+ = P_\mu \,, \qquad [S_5, S_5]_+ = -Z \,, \\
&[G_\mu, Z']_- = -G_\mu \,, \qquad [S_\mu, Z']_- = S_\mu \\
&[S_5, Z']_- = -S_5 \,, \qquad [G_5, Z']_- = G_5 \,, \\
&[Z, Z']_- = -2Z \,, \qquad \left[\tilde{Z}, Z'\right]_- = 2\tilde{Z} \,.
\end{aligned}
\tag{C.4}
$$

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